



# Control of 1-D hyperbolic systems

Jean-Michel Coron

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# Control of 1-D hyperbolic systems

Jean-Michel Coron



Laboratory J.-L. Lions, University Pierre et Marie Curie (Paris 6)  
NetCo 2014

Conference on New Trends in Optimal Control, Tours, June 23-27, 2014



# Outline

- 1 Controllability of 1-D hyperbolic systems
- 2 Dissipative boundary conditions for 1-D hyperbolic systems
- 3 Stabilization of 1-D balance laws
- 4 An open problem: The stabilization of a 1-D water-tank system

- 1 Controllability of 1-D hyperbolic systems
  - The control system
  - Controllability
- 2 Dissipative boundary conditions for 1-D hyperbolic systems
  - The equations
  - Main result
  - Comparaison with prior results
  - Proof of the exponential stability
  - Application to the control of open channels
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  - Balance laws and basic control Lyapunov functions
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# The hyperbolic control system considered

Our hyperbolic control system is

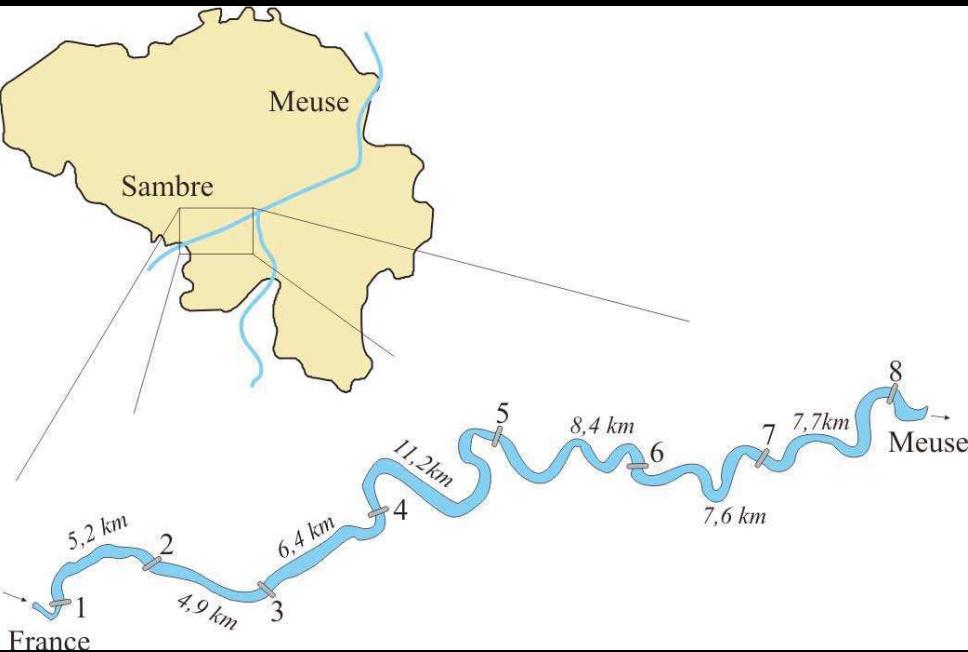
$$(1) \quad y_t + A(y)y_x = 0, \quad (t, x) \in [0, T] \times [0, L],$$

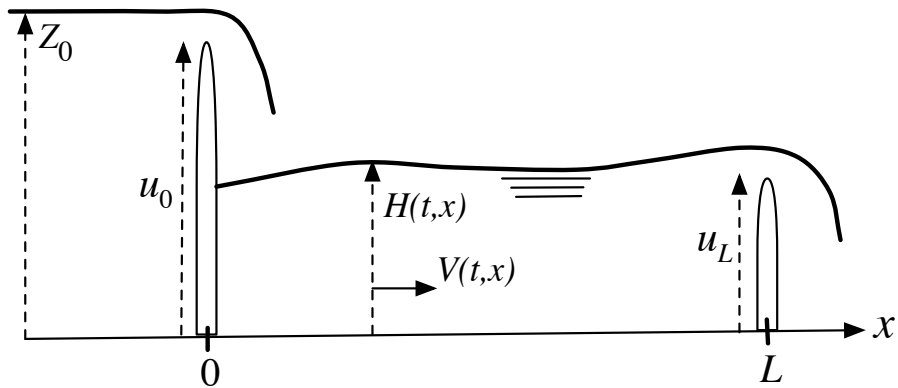
where, at time  $t \in [0, T]$ , the state is  $x \in [0, L] \mapsto y(t, x) \in \mathbb{R}^n$ . Let  $y^* \in \mathbb{R}^n$  be fixed. Assume that  $A(y^*)$  has  $n$  distinct real distinct eigenvalues:  $\Lambda_1(y^*) < \dots < \Lambda_m(y^*) < 0 < \dots < \Lambda_n(y^*)$  for some  $m \in \{0, \dots, n\}$ . After a linear change of variables, we may assume that  $A(y^*) = \text{diag}(\Lambda_1(y^*), \dots, \Lambda_n(y^*))$ . For  $y \in \mathbb{R}^n$ , let  $y_- \in \mathbb{R}^m$  and  $y_+ \in \mathbb{R}^{n-m}$  be such that

$$y = \begin{pmatrix} y_- \\ y_+ \end{pmatrix}.$$

The control is  $y_+(t, 0)$  and  $y_-(t, L)$ .









# The Saint-Venant equations

The index  $j$  is for the  $j$ -th reach.

Conservation of mass:

$$(1) \quad H_{jt} + (H_j V_j)_x = 0.$$

Conservation of momentum:

$$(2) \quad V_{jt} + \left( g H_j + \frac{V_j^2}{2} \right)_x = 0.$$

Flow rate:  $Q_j = H_j V_j$ .



Barré de Saint-Venant  
(Adhémar-Jean-Claude)  
1797-1886

Théorie du mouvement non permanent des eaux, avec applications aux crues des rivières et à l'introduction des marées dans leur lit, C. R. Acad. Sci. Paris Sér. I Math., vol. 53 (1871), pp.147–154.

# Riemann coordinates

Let  $l$  be the number of reaches. Let  $n = 2l$ ,

$$(1) \quad y_i = V_i - \sqrt{2gH_i}, \quad y_{i+l} = V_i + \sqrt{2gH_i}, \quad \forall i \in \{1, \dots, l\},$$

$$(2) \quad \Lambda_i = V_i - \sqrt{gH_i}, \quad \Lambda_{i+l} = V_i + \sqrt{gH_i}, \quad \forall i \in \{1, \dots, l\}.$$

Then the Saint-Venant equations can be written as

$$(3) \quad y_t + A(y)y_x = 0,$$

with

$$(4) \quad A(y) := \text{diag} (\Lambda_1(y), \dots, \Lambda_n(y)).$$

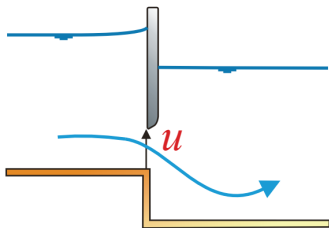
The flow is subcritical flow  $V_i < \sqrt{gH_i}$ ,  $\forall i \in \{1, \dots, l\}$ . For subcritical flows, one has  $m = l$ .

# La Sambre: Hydraulic gates



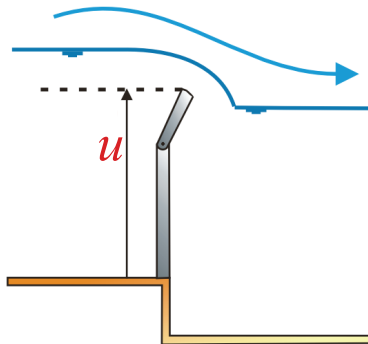
# Boundary conditions

Underflow (sluice)



$$Q = K \sqrt{u(H_{up} - H_{down})}$$

Overflow (spillway)



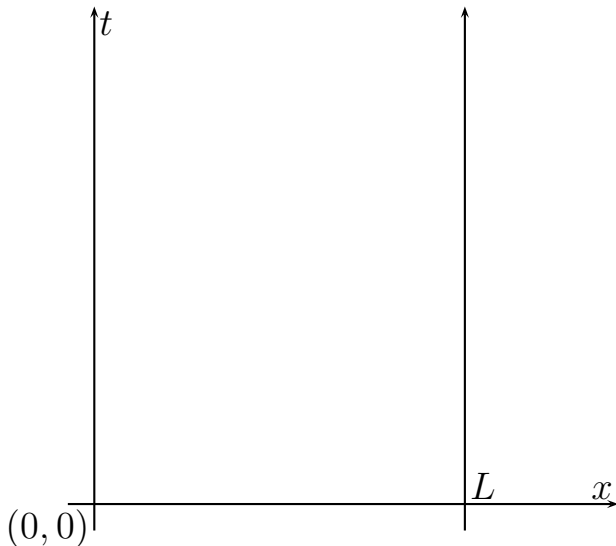
$$Q = K(H_{up} - u)^{3/2}$$

# The control problem

Let  $T > 0$ . Given  $y^0 : [0, 1] \rightarrow \mathbb{R}^n$  and  $y^1 : [0, 1] \rightarrow \mathbb{R}^n$ . Does there exist a solution of the control system such that  $y(0, x) = y^0(x)$  and  $y(T, x) = y^1(x)$ ? Local controllability:  $y^0$  and  $y^1$  are close to  $y^*$ .

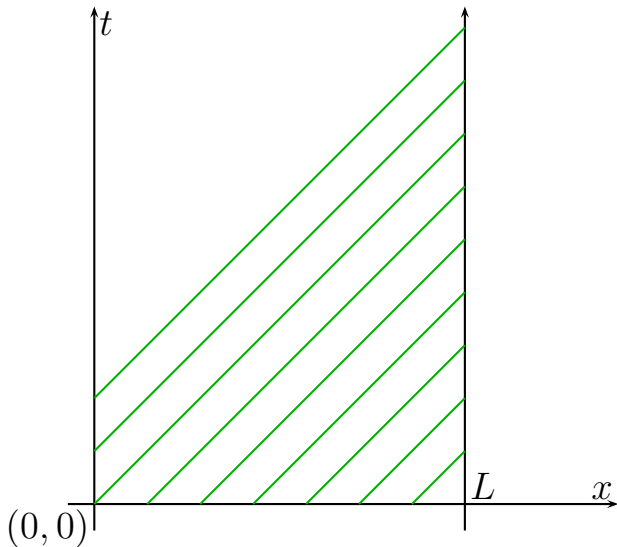
# A very simple example

$$y_t + y_x = 0, \quad y(t, 0) = u(t), \quad x \in [0, L], \quad t > 0.$$



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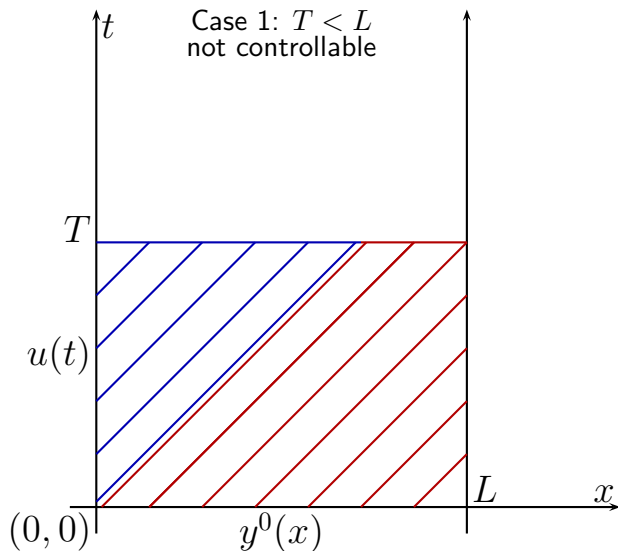
$$y_t + y_x = 0, \quad y(t, 0) = u(t), \quad x \in [0, L], \quad t > 0.$$





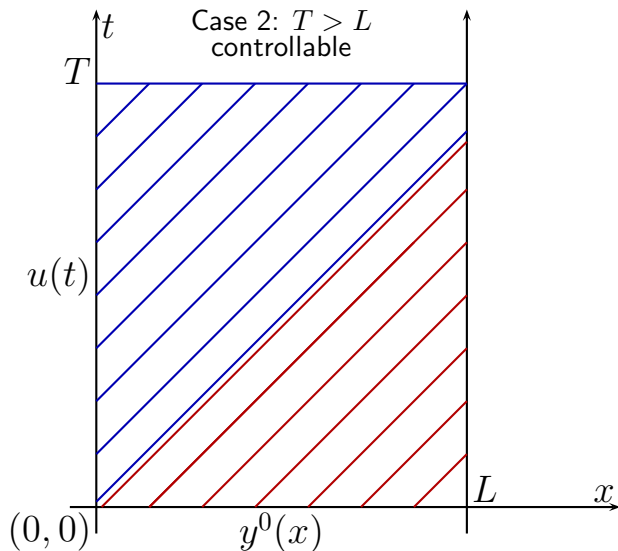
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# Controllability theorem

Theorem (Tatsien Li and Bopeng Rao (2003))

*The local controllability for the  $C^1$ -norm holds if and only if  $T > T_c$  with*

$$(1) \quad T_c := \max \left\{ \frac{L}{|\Lambda_1(y^*)|}, \dots, \frac{L}{|\Lambda_n(y^*)|} \right\}.$$

- For the control on one side only, see T. Li and B. Rao (2002).
- Global steady states controllability for the Saint-Venant equations: M. Gugat (2003), M. Gugat and G. Leugering (2003, 2009). Friction and slopes are allowed in the last paper.
- Generalization:  $A(t, x, y)$ : Z. Wang (2007).

# Complements: BV solutions

- F. Ancona and A. Marson (1998):  $y_t + f(y)_x = 0$ ,  $x \in (0, +\infty)$ ,  $y(0, x) = 0$ : reachable set.
- Th. Horsin (1998):  $y_t + (y^2/2)_x = 0$ ,  $x \in (0, L)$ ,  $y(0, x) = y^0(x)$ : approximate controllability under general conditions on the desired target.
- V. Perrollaz (2012):  $y_t + (f(y))_x = u(t)$ ,  $x \in (0, L)$ ,  $y(0, x) = y^0(x)$ : global controllability in small time.
- Adimurthi, S. Ghoshal and G. Gowda (2014):  $y_t + (f(y))_x = 0$ ,  $x \in (0, L)$ ,  $y(0, x) = y^0(x)$ : reachable set.
- A. Bressan and G. Coclite (2002): In the BV class of entropy solutions, already with  $n = 2$ , there are cases where one cannot steer the control system from  $y^0$  to  $y^*$  even if  $y^0$  is close to  $y^*$  (if one remains close to  $y^*$  in the BV-norm).
- O. Glass: One-dimensional Euler isentropic (2007) and non-isentropic system (2014), both in Eulerian and Lagrangian coordinates: Study of the controllability. Corollary: one can steer the control to  $y^*$  if  $y^0$  is close (in the BV-norm) to  $y^*$  (while remaining close to  $y^*$  in the BV-norm).

# Sketch of proof of Li-Rao's theorem

We assume that

$$(1) \quad T > \max\{L/|\Lambda_i(y^*)|; i \in \{1, \dots, n\}\}.$$

Let  $T_1 > 0$  be such that

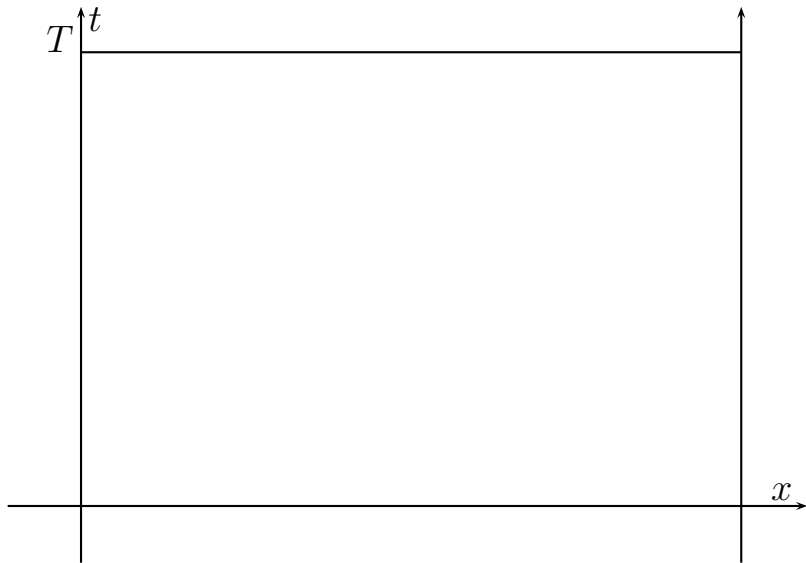
$$(2) \quad T_1 > \frac{1}{2} \max\{L/|\Lambda_i(y^*)|; i \in \{1, \dots, n\}\},$$

$$(3) \quad 2T_1 < T.$$

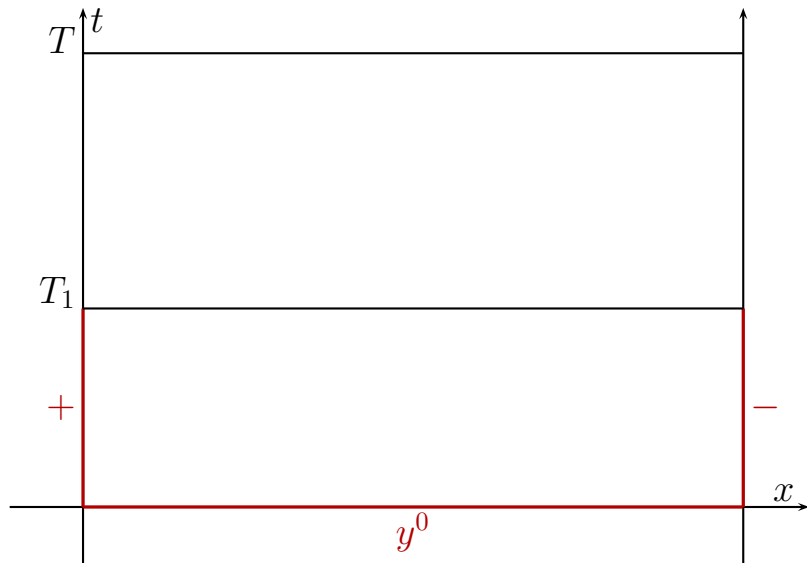
Let us recall that, for  $y \in \mathbb{R}^n$ ,  $y_- \in \mathbb{R}^m$  and  $y_+ \in \mathbb{R}^{n-m}$  are such that

$$y = \begin{pmatrix} y_- \\ y_+ \end{pmatrix}.$$

## Sketch of proof (continued)

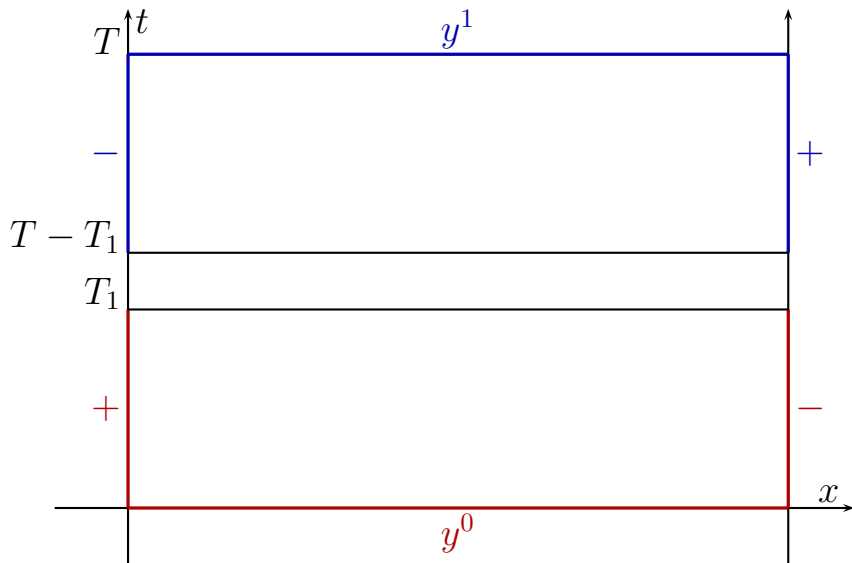


## Sketch of proof (continued)

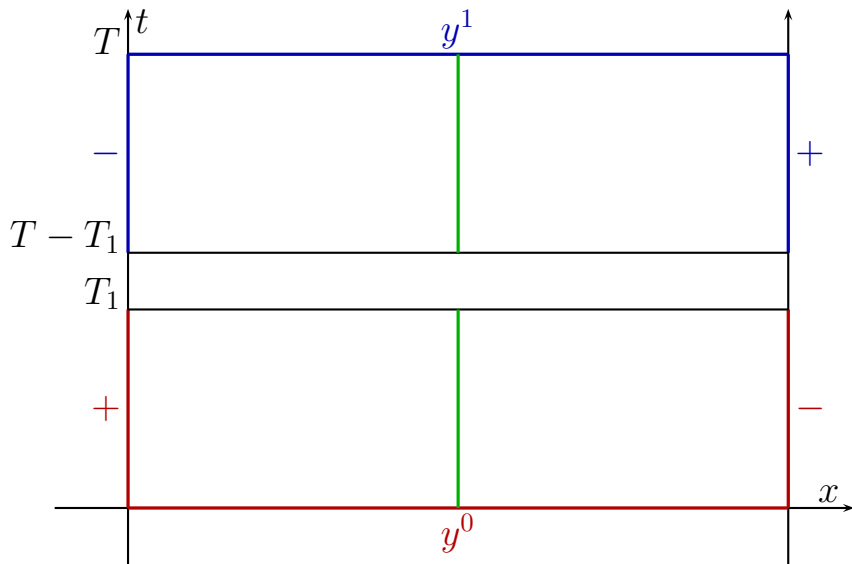




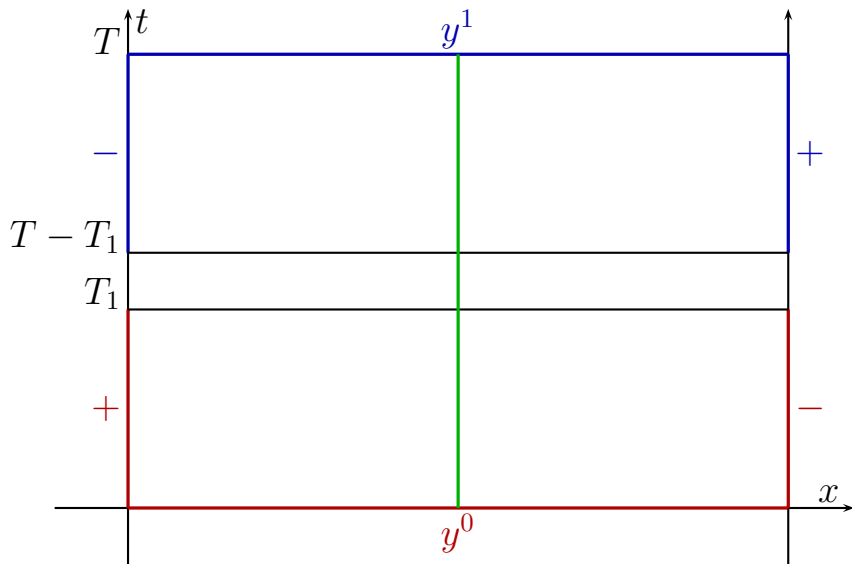
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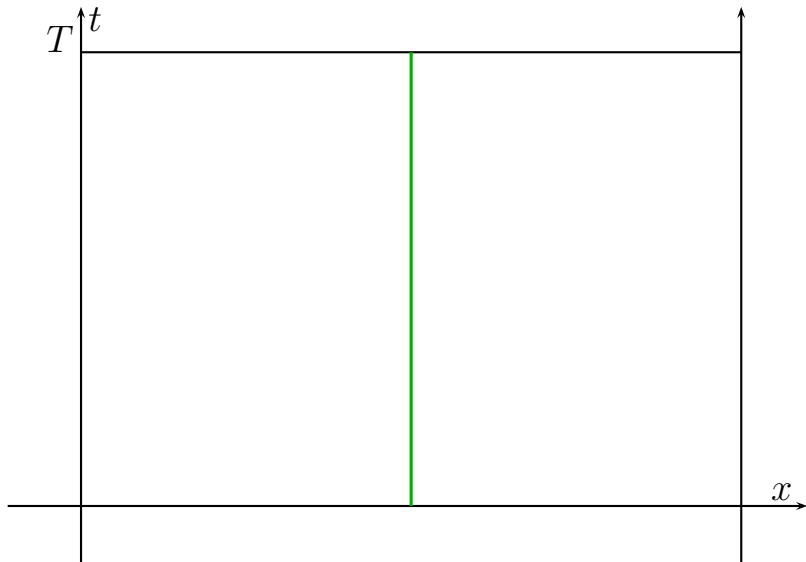
## Sketch of proof (continued)



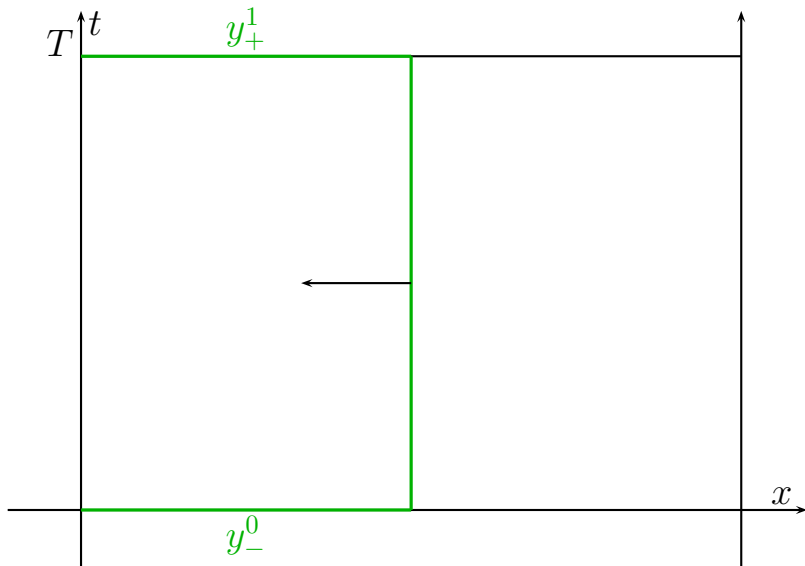
## Sketch of proof (continued)



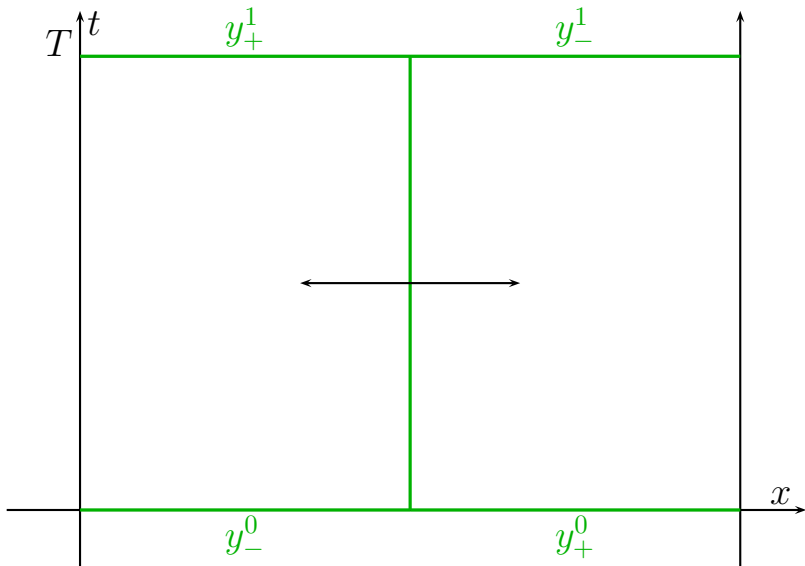
## Sketch of proof (continued)



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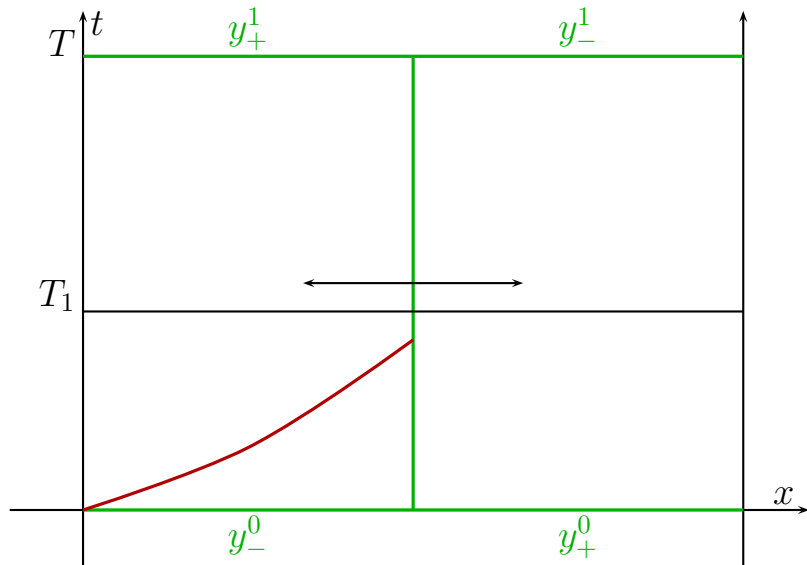


## Sketch of proof (continued)



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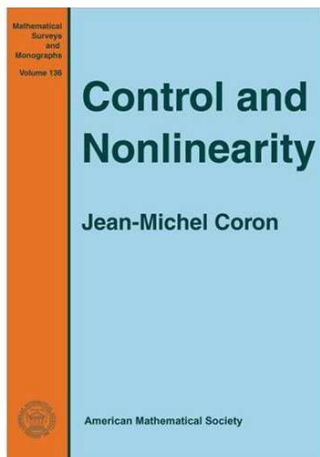
$$T_1 > \max\{L/|\Lambda_i(0)|; i \in \{1, \dots, n\}\}/2.$$



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# More on controllability and stabilization



JMC, Control and nonlinearity,  
Mathematical Surveys and  
Monographs, 136, 2007, 427 p. Pdf  
file freely available from my web  
page.

# Double inverted pendulum (CAS, ENSMP/La Villette)

